Reg. No. : ....

Code No. : 5673 **Sub. Code : ZMAM 23**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2022

Second Semester

Mathematics — Core

ADVANCED CALCULUS

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let f and g be continuous and bounded on D, then If $F(p) \geq 0$ for all $p \in D$, $\iint_D f$ ———— 0.

   (a) $=$ (b) $>$

   (c) $\geq$ (d) $\leq$

2. If f is continuous on $R,$ then $\lim_{d(N)\downarrow 0}|\overline{S}(N)-\underline{S}(N)|$ ———— 0.

(a) $\neq$ (b) $=$

(c) $>$ (d) $<$

3. The linear function L such that $L(1,0,0)$, $(0,1,0)$, $(0,0,1)$ is ————

(a) [2, 1, 3] (b) [–2, 1, 3]

(c) [2, –1, –3] (d) [2, –1, 3]

4. The differentials of the following transformations at the indicated points

$$\begin{cases} u &= x+6y \\ v &= 3xy \\ w &= x^2-3y^2 \end{cases} \quad \text{at } (1,\,1) \text{ is} \text{ ————}$$

(a) $\begin{bmatrix} 1 & 6 \\ 3 & 3 \\ -2 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 \\ 3. & 3 \\ 2 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 6 \\ 3 & 3 \\ 2 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -6 \\ 3 & 3 \\ 2 & 6 \end{bmatrix}$

(b) Prove that let R be a cube in (x, y, z) space with faces parallel to the coordinate planes. Let ω be a 2-form $\omega = A\,dydz + B\,dzdx + C\,dxdy$.

———————

5. Find the product of the matrices $\begin{bmatrix} \cos y & \sin y \\ -x^{-1}\sin y & x^{-1}\cos y \end{bmatrix} \begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{bmatrix}$ is ————

(a) 1 (b) $2I$

(c) 0 (d) $-I$

6. Find the det of $\begin{bmatrix} 8 & 2 \\ 12 & 3 \end{bmatrix}$ is ————

(a) 24 (b) –12

(c) 0 (d) 12

7. If $T: \begin{cases} x = u + v \\ y = v - u^2 \end{cases}$ then the Jacobian is ————

(a) $1-2u$ (b) $1+2u$

(c) $1+2v$ (d) $1-2v$

8. If E is a closed bounded subset of Ω of zero volume, then $T(E)$ has ———— volume.

(a) 0 (b) 1

(c) –1 (d) ∞

9. If f is a scalar function of class C'', then $curl\ (grad(f)) =$

(a) 0 (b) 1

(c) -1 (d) ∞

10. If ω is any differential form of class C'', then $dd\omega =$ ————

(a) 0 (b) 1

(c) ∞ (d) -1

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let f and f_1 be defined and continuous for $x \in [a, b]$, $y \in [c, d]$ and F defined by $f(x) = \int_c^d f(x, y)\, dy$. Then, prove that $F'(x)$ exists on the interval $[a, b]$ and is given by $F'(x) = \int_c^d \frac{\partial f}{\partial x} dy = \int_c^d f_1(x, y)\, dy$.

Or

(b) Let B be the closed ball in n space, center 0, radius r, let T be a C' transformation defined on an open set containing B on which its Jacobian $J(p)$ never vanishes Suppose also that T is closed to the identify map, meaning that there is a number ρ such that $0 < \rho < \frac{1}{2}$ and $|T(p) - p| \leq pr$ for all $\rho \in \beta$.

Then, prove that T maps B onto a set $T(B)$ that contains all the points in the open ball centered at 0 of radius $(1 - 2\rho)r$.

20. (a) Prove that let D be a closed convex region in the plane, and let $\omega = A(x, y)dx + B(x, y)dy$ with A and B of class C' and D. then, $\int_{\partial D} A\, dx + B\, dy = \iint_D d\omega = \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dxdy$.

Or

(b) Prove that let T be continuous on a set D. then, any compact set $C \subset D$ is carried by T into a compact set $T(C)$, and any connected set $S \subset D$ is carried into a connected set $T(S)$.

(a) Prove that let T be of class C' in an open set D, with $J(p) \neq 0$ for all $p \in D$. Suppose also that T is globally 1-to-1 in D, so that there is an inverse transformation T^{-1} defined on the set $T(D) = D^*$. Then, T^{-1} is of class C' on D^*, $d(T^{-1})|_q = (dT|_p)^{-1}$, where $q = T(p)$.

Or

(b) If u, v and w are C' functions of x, y, and z in D, and if $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ at all points of D, then u, v and w are functionally related in D. Find this relationship.

(a) Let T be a transformation from R^2 into R^2 which is of class C' in an open region D. furthermore, let T be conformal and have a strictly positive Jacobian throughout D. Then, prove that at each point of D, the differential of T has a matrix representation of the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$.

Or

(b) If f is continuous on R, then prove that $\lim_{d(N)\downarrow 0} \left|\overline{S}(N) - \underline{S}(N)\right| = 0$.

12. (a) Let the transformation S be continuous on a set A and T be continuous on a set B, and let $p_0 \in A$ and $S(p_0) = q_0 \in B$. Then, prove that the product transformation TS, defined by $TS(p) = T(S(p))$, is continuous at p_0.

Or

(b) Compute the rank of matrix $\begin{bmatrix} 3 & -6 & 9 \\ 2 & -4 & 6 \\ -2 & 4 & -12 \end{bmatrix}$.

13. (a) Compute the Jacobians transformation $\begin{cases} u = e^x \cos y \\ v = e^x \sin y. \end{cases}$

Or

(b) Prove that, if T is continuous and 1-to-1 on a compact set D, then T has a unique inverse T^{-1} which maps $T(D) = D^*$ 1-to-1 onto D, and T^{-1} is continuous on D^*, for, the graph of T^{-1} is just the reflection of the graph of T and is also compact, so that the transformation T^{-1} must also be continuous.

14. (a) If E is a closed bounded subset of Ω of zero volume, then prove that $T(E)$ has zero volume.

Or

(b) If γ_1 and γ_2 are smoothly equivalent curves, then prove that $L(\gamma_1) = L(\gamma_2)$.

15. (a) If ω is any differential form of class C'', then prove that $dd\omega = 0$.

Or

(b) Prove that let T be a transformation of class C''' defined by $x = \phi(u, v)$, $y = \psi(u,v)$, mapping a compact set D onto D^*. we assume that D and D^* are finite unions of standard region and that T is 1-to-1 on the boundary of D and maps it onto the boundary of D^*. let f be continuous in D^*. then $\iint_{D^*} f(x,y)\,dxdy = \iint_D f(\phi(u,v), \psi(u,v))$ $\frac{\partial(x,y)}{\partial(u,v)} dudv$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let ϕ' exist and be continuous on the interval $[\alpha, \beta]$ with $\phi(\alpha) = 0$ and $\phi(\beta) = b$. Let f be continuous at all points $\phi(u)$ for $\alpha \le u \le \beta$. Then, prove that $\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(u))\phi'(u)du$.

Or

(b) Prove that let R be a closed rectangle, and let f be bounded in R and continuous at all points of R except those in a set E of zero area. Then $\iint_R f$ exists.

17. (a) Prove that let T be differentiable on an open set D, and let S be differentiable on D, and if $p \in D$ and $q = T(p)$, then prove that $d(ST)|_p = dS|_q \, dT|_p$.

Or